ON SOME RECENT RESULTS IN THE THEORY OF THE ZETA-FUNCTION

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1. Introduction

This review article is devoted to the Riemann zeta-function $\zeta(s)$, defined for $\Re s > 1$ as

$$(1.1) \qquad \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \,:\, \text{prime}} (1 - p^{-s})^{-1},$$

and otherwise by analytic continuation. It admits meromorphic continuation to the whole complex plane, its only singularity being the simple pole $s = 1$ with residue 1. For general information on $\zeta(s)$ the reader is referred to the monographs [13], [16], [21], [28] and [51]. From the functional equation

$$(1.2) \qquad \zeta(s) = \chi(s)\zeta(1 - s), \quad \chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1 - s),$$

which is valid for any complex s, it follows that $\zeta(s)$ has zeros at $s = -2, -4, \ldots$. These zeros are called the "trivial" zeros of $\zeta(s)$, to distinguish them from the complex zeros of $\zeta(s)$. The zeta-function has also an infinity of complex zeros. It is well-known that all complex zeros of $\zeta(s)$ lie in the so-called "critical strip" $0 < \sigma = \Re s < 1$, and if $N(T)$ denotes the number of zeros $\rho = \beta + i\gamma$ (β, γ real) of $\zeta(s)$ for which $0 < \gamma \le T$, then ($f = O(g)$ and $f \ll g$ both mean that $|f(x)| \le Cg(x)$ for some $C > 0$ and $x \ge x_0$)

$$(1.3) \qquad N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right)$$

with

$$(1.4) \qquad S(T) = \frac{1}{\pi}\arg\zeta(\tfrac{1}{2} + iT) = O(\log T).$$

This is the so-called Riemann–von Mangoldt formula. *The Riemann hypothesis* (henceforth RH for short) is the conjecture, stated by B. Riemann in his epoch-making memoir [49], that *very likely – sehr wahrscheinlich – all complex zeros of $\zeta(s)$ have real parts equal to* $\frac{1}{2}$. For this reason the line $\sigma = \frac{1}{2}$ is called the "critical line" in the theory of $\zeta(s)$. The RH is undoubtedly one of the most celebrated and difficult open problems in whole Mathematics. Its proof would have very important consequences in multiplicative number theory, especially in problems involving the distribution of primes, since the defining relation (1.1) shows

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the intrinsic connection between $\zeta(s)$ and primes. It would also very likely lead to generalizations to many other zeta-functions (Dirichlet series) sharing similar properties with $\zeta(s)$. Despite much evidence in favour of the RH, there are also some reasons to be skeptical about its truth – see, for example, [24], [27] and [30]. The aim of this paper is to present briefly some recent results in the theory of $\zeta(s)$. The choice of subjects is motivated by the limited length of this text and by the author's personal research interests. An extensive bibliography will aid the interested reader.

2. ZEROS ON THE CRITICAL LINE

The distribution of the zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ on the critical line $\sigma = \frac{1}{2}$ is a fundamental problem in the theory of $\zeta(s)$. Despite enormous efforts the RH has been so far neither proved nor disproved. The smallest zeros of $\zeta(s)$ (in absolute value) are $\frac{1}{2} \pm 14.134725\ldots i$. Large scale computations of zeros of $\zeta(s)$ have been carried out in recent times by the aid of computers (see [44]–[46], [48]). Suffice to say that it is known today that the first 1.5 billion complex zeros of $\zeta(s)$ in the upper half-plane are simple and do have real parts equal to $\frac{1}{2}$, as predicted by the RH. Moreover, many large blocks of zeros of much greater height have been thoroughly investigated, and all known zeros satisfy the RH. The RH is, however, not the only topic connected with the zeros on the critical line. Let $N_0(T)$ denote the number of zeros of $\zeta(s)$ of the form $\frac{1}{2} + it, 0 < t \le T$. The RH is in fact the statement that $N_0(T) = N(T)$ for $T > 0$. A fundamental result of A. Selberg [50] from 1942 states that

$$(2.1) \qquad\qquad N_0(T) > CN(T)$$

for $T \ge T_0$ and some positive constant $C > 0$. In other words, a positive proportion of all complex zeros of $\zeta(s)$ lies on the critical line. A substantial advancement in this topic was made by N. Levinson [39] in 1974, who proved that one can take $C = 1/3$ in (2.1). By refining Levinson's techniques further improvements have been made, and today it is known that one can take $C = 2/5$ in (2.1), but there is no chance that by existing methods one can reach the value $C = 1$.

Another interesting problem is the problem of the gap between consecutive zeros on the critical line. Let us denote by $0 < \gamma_1 \le \gamma_2 \le \ldots$ the positive zeros of $\zeta(\frac{1}{2} + it)$ with multiplicities counted. If the RH is true, then it is known (see [51]) that the function $S(T)$, defined by (1.3) and (1.4), satisfies

$$(2.2) \qquad\qquad S(T) = O\Big(\frac{\log T}{\log\log T}\Big).$$

This seemingly small improvement over (1.4) is significant: If (2.2) holds, then from (1.3) one infers that $N(T + H) - N(T) > 0$ for $H = C/\log\log T$ with a suitable $C > 0$ and $T \ge T_0$. Consequently we have, assuming the RH, the bound

$$(2.3) \qquad\qquad \gamma_{n+1} - \gamma_n \ll \frac{1}{\log\log \gamma_n}$$

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for the gap between consecutive zeros on the critical line. The bound (2.3) is certainly out of reach at present. For some unconditional results on $\gamma_{n+1} - \gamma_n$, see [17]–[19] and [33]. For example, it was proved by the author [16], [17] that one has unconditionally

$$\gamma_{n+1} - \gamma_n \ll \gamma_n^{\theta+\varepsilon}, \qquad \theta = \frac{\kappa + \lambda}{4(\kappa + \lambda) + 2},$$

where (κ, λ) is a so-called "exponent pair" from the theory of exponential sums (see [8], [16] for definition), and ε denotes arbitrarily small positive constants. With known results on exponent pairs one obtains then as the strongest known result $\theta \leq 0.155945\ldots$. Another unconditional result of the author (see [19]) states that

$$\sum_{\gamma_n \leq T} (\gamma_{n+1} - \gamma_n)^3 \ll T \log^6 T.$$

3. Zero-density results

In the absence of a proof of RH, in practice one has to assume the existence of the zeros of $\zeta(s)$ in $\frac{1}{2} < \sigma < 1$ and show that the influence of these zeros is not large. One defines the zero-counting function

$$N(\sigma, T) = \sum_{\zeta(\beta+i\gamma)=0, \beta \geq \sigma, |\gamma| \leq T} 1$$

and tries to obtain upper bounds for $N(\sigma, T)$. By the symmetry of zeros it is sufficient to assume that $\sigma \geq \frac{1}{2}$, and one also often takes into account the so-called "zero-free region" of $\zeta(s)$. This is due to I.M. Vinogradov (see Ch. 6 of [16]) and says that

(3.1)
$$\beta \leq 1 - C(\log \gamma)^{-2/3}(\log \log \gamma)^{-1/3} \qquad (\zeta(\beta + i\gamma) = 0, \ \gamma \geq \gamma_0 > 0, \ C > 0).$$

This result was obtained in 1958, and is one of the oldest results in zeta-function theory which has not been subsequently improved.

In obtaining upper bounds for $N(\sigma, T)$ one can use several techniques (see Ch. 11 of [16] for a comprehensive account). In the range $\frac{1}{2} < \sigma \leq \frac{3}{4}$ the best bound is due to A.E. Ingham [10]. This is

(3.2)
$$N(\sigma, T) \ll T^{3(1-\sigma)/(2-\sigma)} \log^5 T,$$

while in the range $\frac{3}{4} \leq \sigma \leq 1$ one has M.N. Huxley's result [6] that

(3.3)
$$N(\sigma, T) \ll T^{3(1-\sigma)/(3\sigma-1)} \log^{44} T.$$

When combined, (3.2) and (3.3) yield

$$(3.4) \qquad N(\sigma, T) \ll T^{12(1-\sigma)/5} \log^{44} T \qquad (\tfrac{1}{2} \le \sigma < 1).$$

From (3.4) one can deduce the bound

$$p_{n+1} - p_n \ll p_n^{7/12} \log^C p_n,$$

where p_n is the n–th prime (see Ch. 12 of [16]). By combining analytic and sieve techniques various authors have reduced the exponent "7/12", and the current record holders are Baker and Harman [2], who proved that

$$p_{n+1} - p_n \ll p_n^{0.535}.$$

It is interesting that the so-called "density hypothesis"

$$(3.5) \qquad N(\sigma, T) \ll T^{2-2\sigma+\varepsilon} \qquad (\tfrac{1}{2} \le \sigma \le 1)$$

gives the bound

$$p_{n+1} - p_n \ll p_n^{\frac{1}{2}+\varepsilon},$$

while the much stronger RH gives only a slightly better result, namely

$$p_{n+1} - p_n \ll p_n^{\frac{1}{2}} \log p_n,$$

which is still insufficient to prove the old conjecture: *between every two squares there is always a prime.*

The bound (3.5) is known to hold for $\sigma \ge \frac{11}{14}$, which was proved by M. Jutila [38] in 1977. For various values of σ in the range $\frac{3}{4} < \sigma < 1$ sharper results than (3.3) or (3.4) have been proved. For example, the author [11], [14]–[16] obtained the bounds

$$N(\sigma, T) \ll T^{3(1-\sigma)/(2\sigma)+\varepsilon} \qquad (\tfrac{3831}{4791} = 0.799624\ldots \le \sigma \le 1),$$
$$N(\sigma, T) \ll T^{9(1-\sigma)/(7\sigma-1)+\varepsilon} \qquad (\tfrac{41}{53} = 0.773585\ldots \le \sigma \le 1),$$
$$N(\sigma, T) \ll T^{6(1-\sigma)/(5\sigma-1)+\varepsilon} \qquad (\tfrac{13}{17} = 0.764705\ldots \le \sigma \le 1).$$

4. THE MEAN SQUARE

Power moments, and especially the mean square formulas, play an important rôle in the theory of $\zeta(s)$. In fact, zero-density estimates discussed in the previous section depend heavily on power moments of $\zeta(s)$, as discussed extensively in Chapter 11 of [16]. As far as mean square formulas are concerned, one can distinguish between the cases $\sigma = \frac{1}{2}$ and $\frac{1}{2} < \sigma \le 1$. Here we shall briefly discuss only the former

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case, and for the latter we refer the reader to [34], [41]. One has the asymptotic formula

$$(4.1) \qquad \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t \; = \; T \log\left(\frac{T}{2\pi}\right) + (2\gamma - 1)T + E(T),$$

where $\gamma = 0.577\ldots$ is Euler's constant, and $E(T)$ is to be considered as the error term in the asymptotic formula (4.1). F.V. Atkinson [1] established in 1949 an explicit, albeit complicated formula for $E(T)$, containing two exponential sums of length $\asymp T$, plus an error term which is $O(\log^2 T)$. Atkinson's formula has been the starting point for many results on $E(T)$. It is conjectured that $E(T) \ll T^{1/4+\varepsilon}$, and this bound cannot be proved even if the RH is assumed. The best known upper bound for $E(T)$, obtained by intricate estimation of a certain exponential sum, is due to M.N. Huxley [7]. This is

$$E(T) \; \ll \; T^{72/227}(\log T)^{679/227}, \qquad \frac{72}{227} = 0.3171806\ldots\,.$$

In the other direction, J.L. Hafner and the author [3], [21] proved in 1987 that there exist absolute constants $A, B > 0$ such that

$$E(T) = \Omega_+ \left\{ (T \log T)^{1/4} (\log\log T)^{(3+\log 4)/4} e^{-A\sqrt{\log\log\log T}} \right\}$$

and

$$E(T) = \Omega_- \left\{ T^{1/4} \exp\left(\frac{B(\log\log T)^{1/4}}{(\log\log\log T)^{3/4}} \right) \right\}.$$

The omega-symbols are customarily defined as follows: $f = \Omega(g)$ means that $f = o(g)$ does not hold, $f = \Omega_+$ means that $\limsup f/g > 0$, $f = \Omega_-$ means that $\liminf f/g < 0$, and $f = \Omega_\pm(g)$ means that $\limsup f/g > 0$ and $\liminf f/g < 0$ both hold. A quantitative Ω–result for $E(T)$ was proved by the author [23]: There exist constants $C, D > 0$ such that for $T \geq T_0$ every interval $[T, T + CT^{1/2}]$ contains points t_1, t_2 for which

$$E(t_1) > Dt_1^{1/4}, \qquad E(t_2) < -Dt_2^{1/4}.$$

Numerical calculations pertaining to $E(T)$ have been carried out in [37] by the author and H. te Riele. Power moments of $E(T)$ were considered in 1983 by the author [12], where it was proved that

$$\int_0^T |E(t)|^A \, \mathrm{d}t \; \ll \; T^{(A+4)/4+\varepsilon} \qquad (0 \leq A \leq \tfrac{35}{4}),$$

which in view of the Ω–results is (up to "ε") optimal, and supports the conjecture $E(T) \ll T^{1/4+\varepsilon}$. Later in 1992 D.R. Heath-Brown [5] used these bounds to investigate the distribution function connected with $E(T)$. In the special case $A = 2$ it is known that

$$(4.2) \quad \int_0^T E^2(t)\,\mathrm{d}t \;=\; \frac{2C}{3}(2\pi)^{-1/2}T^{3/2} + O(T\log^4 T), \qquad C = \sum_{n=1}^\infty d^2(n)n^{-3/2}$$

where $d(n)$ is the number of divisors of n. The bound for the error term in (4.2) was obtained independently by E. Preissmann [47] and the author [21].

In 1992 K.-M. Tsang [52] proved that, for some $\delta > 0$ and explicit constants $c_1, c_2 > 0$,

$$\int_0^T E^3(t)\,\mathrm{d}t \;=\; c_1 T^{\frac{7}{4}} + O(T^{\frac{7}{4}-\delta}),$$

$$\int_0^T E^4(t)\,\mathrm{d}t \;=\; c_2 T^2 + O(T^{2-\delta}).$$

The author [31] recently improved these bounds to $O(T^{\frac{47}{28}+\varepsilon})$ and $O(T^{\frac{45}{23}+\varepsilon})$, respectively.

5. The mean fourth power

The asymptotic formula for the fourth moment of the Riemann zeta-function $\zeta(s)$ on the critical line is customarily written as

$$(5.1) \quad \int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t \;=\; TP_4(\log T) \;+\; E_2(T), \; P_4(x) = \sum_{j=0}^4 a_j x^j.$$

A classical result of A.E. Ingham [9] (see Ch. 5 of [16] for a relatively simple proof) is that $a_4 = 1/(2\pi^2)$ and that the error term $E_2(T)$ in (5.1) satisfies the bound $E_2(T) \ll T\log^3 T$. In 1979 D.R. Heath-Brown [4] made significant progress in this problem by proving that $E_2(T) \ll T^{7/8+\varepsilon}$. He also calculated

$$a_3 \;=\; 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2})\pi^{-2}$$

and produced more complicated expressions for a_0, a_1 and a_2 in (5.2). For an explicit evaluation of the a_j's in (5.1) the reader is referred to the author's work [27]. At present the bound

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^k\,\mathrm{d}t \;\ll_\varepsilon\; T^{1+\varepsilon}$$

is not known to hold (see Chs. 7-8 of [16], and [21]) for any constant $k > 4$, which makes the function $E_2(T)$ particularly important in the theory of mean values of

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$\zeta(s)$. In recent years, due primarily to the application of powerful methods of spectral theory (see Y. Motohashi's monograph [49] for a comprehensive account), much advance has been made in connection with $E_2(T)$. This involves primarily exponential sums involving the quantities κ_j and $\alpha_j H_j^3(\frac{1}{2})$. Here as usual $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$ is the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2,\mathbb{Z})$–automorphic forms, and $\alpha_j = |\rho_j(1)|^2(\cosh \pi\kappa_j)^{-1}$, where $\rho_j(1)$ is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue λ_j to which the Hecke series $H_j(s)$ is attached. It is conjectured that $E_2(T) \ll T^{1/2+\varepsilon}$, which would imply the (hitherto unproved) bound $\zeta(\frac{1}{2}+it) \ll t^{1/8+\varepsilon}$. It is known now that

$$(5.2) \qquad E_2(T) = O(T^{2/3}\log^{C_1} T), \quad E_2(T) = \Omega(T^{1/2}),$$

$$(5.3) \qquad \int_0^T E_2(t)\,\mathrm{d}t = O(T^{3/2}), \quad \int_0^T E_2^2(t)\,\mathrm{d}t = O(T^2\log^{C_2} T),$$

with effective constants $C_1, C_2 > 0$ (the values $C_1 = 8, C_2 = 22$ are worked out in [43]). The above results were proved by Y. Motohashi and the author: (5.2) and the first bound in (5.3) in [21],[36] and the second upper bound in (5.3) in [35]. The Ω–result in (5.2) was improved to $E_2(T) = \Omega_\pm(T^{1/2})$ by Y. Motohashi [42]. Recently the author [23] made further progress in this problem by proving the following quantitative omega-result: there exist two constants $A > 0$, $B > 1$ such that for $T \geq T_0 > 0$ every interval $[T, BT]$ contains points T_1, T_2 for which

$$(5.4) \qquad E_2(T_1) > AT_1^{1/2}, \quad E_2(T_2) < -AT_2^{1/2}.$$

Very likely this integral is $\sim CT^2$ for some $C > 0$ as $T \to \infty$. The latest result, proved by the author [32], complements the upper bound in (5.3) for the mean square, and says that

$$\int_0^T E_2^2(t)\,\mathrm{d}t \gg T^2.$$

Very likely this integral is $\sim CT^2$ for some $C > 0$ as $T \to \infty$. In concluding, let it be mentioned that the sixth moment was investigated in [29], where it was shown that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^6\,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}$$

does hold if a certain conjecture involving the so-called ternary additive divisor problem is true. On the other hand, it is known that (see Ch. 9 of [16]) that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \gg_k T(\log T)^{k^2} \qquad (k \in \mathbb{N}).$$

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